Separating the Expressive Power of Propositional Dynamic and Modal Fixpoint Logics

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University of Kassel, Germany

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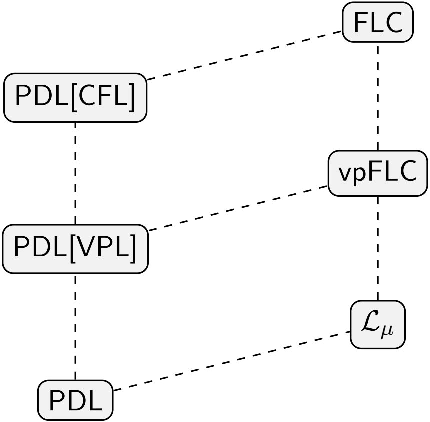
Overview



- All these logics have different expressive power
- Constructive proofs: exhibit property that cannot be expressed
- Dashed lines show proper inclusions

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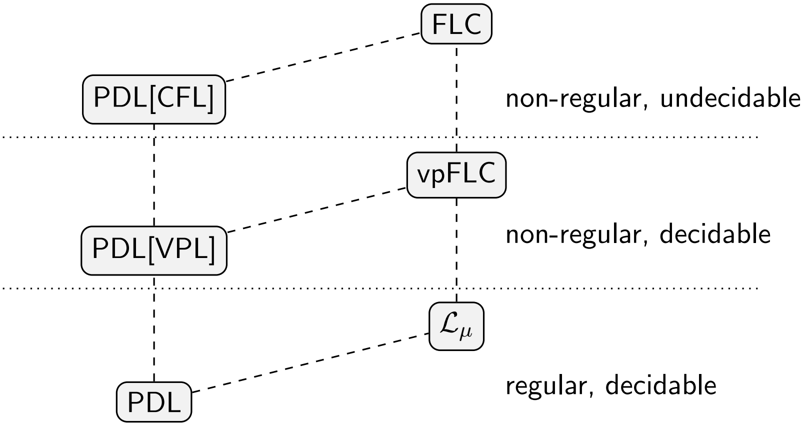

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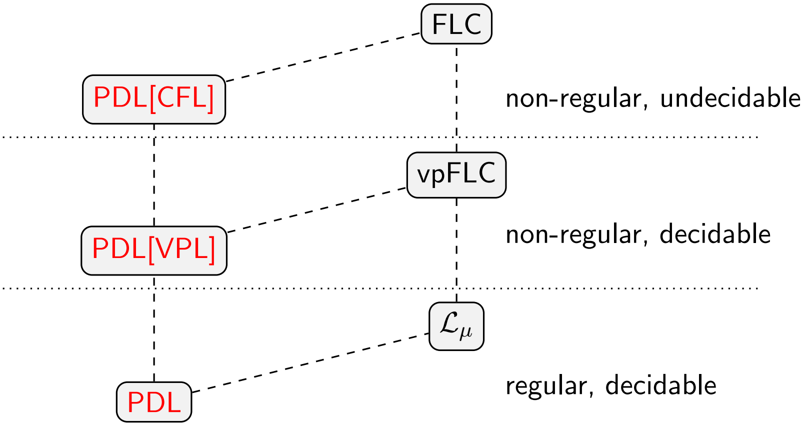

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- Constructive proofs: exhibit property that cannot be expressed
- Dashed lines show proper inclusions

Repetition: Propositional Dynamic Logics

LTS with Propositions $p, q, \ldots$, transitions in $\Sigma = \{a, b, c, \ldots\}$

Formulas:

$$\varphi ::= p \mid \varphi \vee \varphi \mid \neg\varphi \mid \langle L\rangle\varphi$$

$s \in [\![\langle L\rangle\varphi]\!]$ iff ex. $t \in [\![\varphi]\!]$ s.t. $s \xrightarrow{w} t$ w. $w \in L$

- PDL[REG]: L regular over Σ, e.g., $\langle\{a^{3n} \mid n \in \mathbb{N}\}\rangle p$

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- PDL[CFL]: L context-free over Σ, e.g., $\langle a^n b a^n \rangle p$
- PDL[VPL]: L visibly pushdown over Σ, e.g., $\langle a^n b^n \rangle p$
  - visibly pushdown language: accepted by pushdown automaton with stack operations tied to alphabet
  - e.g., must push on a, pop on b, leave stack height be on c
  - visibly pushdown: $\{a^n b^n \mid n \in \mathbb{N}\}$
  - not visibly pushdown: $\{a^n b a^n \mid n \in \mathbb{N}\}$

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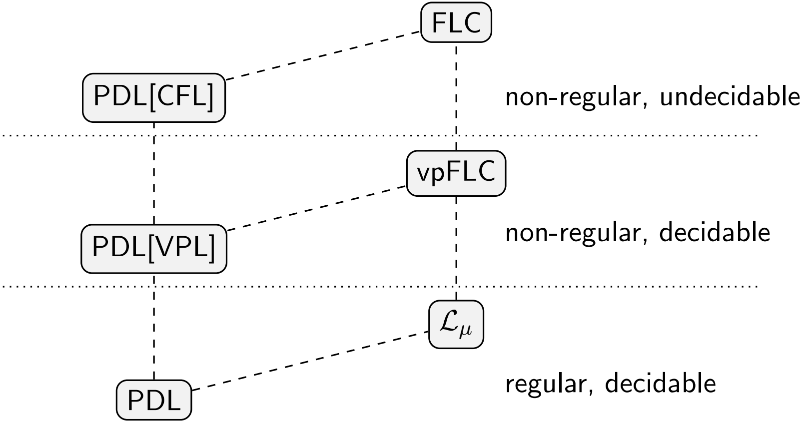
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$\mathcal{L}_\mu$
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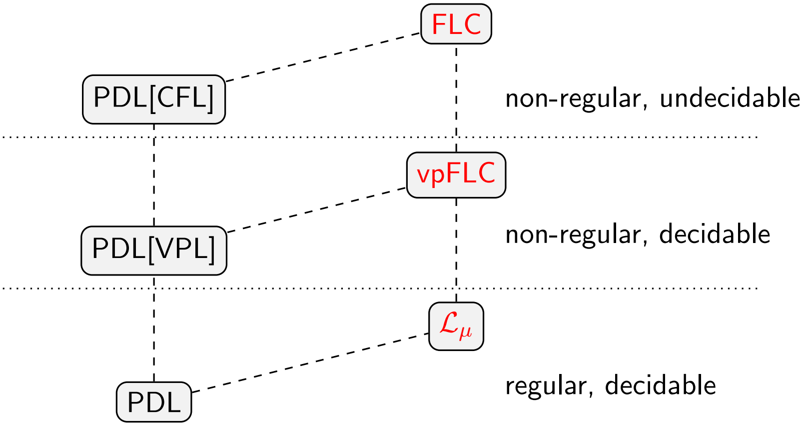
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Idea: Fixpoint unfolding

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- Fixpoint Logic with Chop (FLC):

  $$\varphi ::= p \mid \varphi \vee \varphi \mid \neg\varphi \mid \langle a \rangle \varphi \mid X \mid \mu X.\varphi \mid \varphi ; \varphi \mid \tau$$

  Work behind recursion anchor, ";" is concatenation

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&= \bigvee_{n\in\mathbb{N}} \langle a^n\rangle; [b]; \langle a^n\rangle; p
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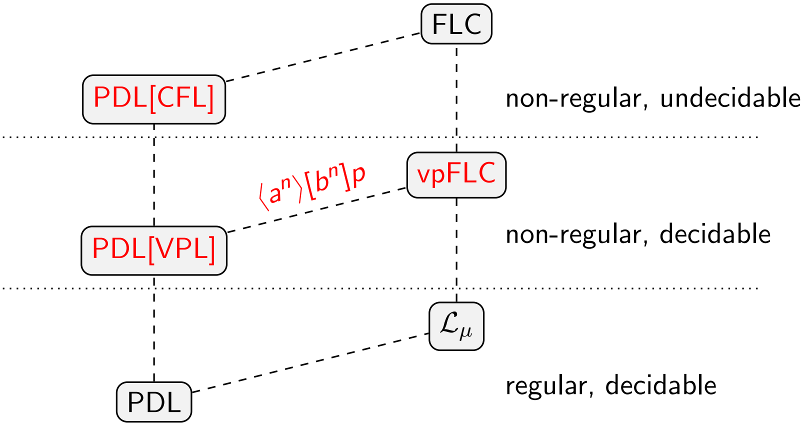
Formula: $\langle a^n\rangle[b]\langle a^n\rangle; p$

- Visibly pushdown FLC (vpFLC):
  - constraints on concatenation of modal operators, mimic VPL
  - Dedicated talk: CONCUR session Thursday 26th, 14:00–15:15
  - allowed: $(\mu X.\langle a\rangle;[b] \vee \langle a\rangle;X;[b]);p$ "=" $\langle a^n\rangle[b^n]p$
  - not allowed: $(\mu X.[b] \vee \langle a\rangle;X;\langle a\rangle);p$ "=" $\langle a^n\rangle[b]\langle a^n\rangle p$ due to VPL constraints

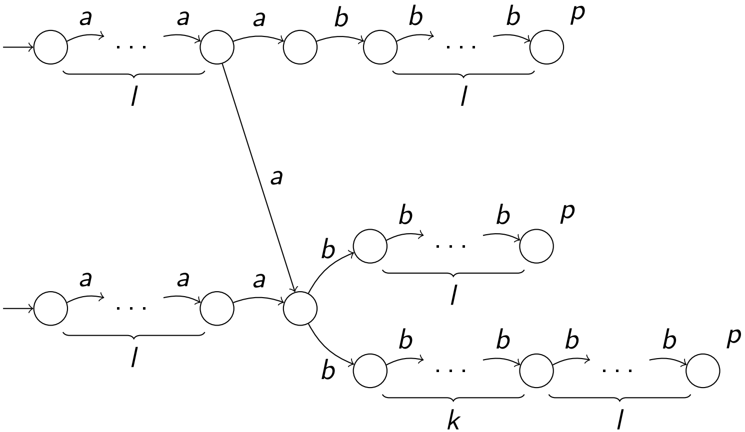
Overview



Separate PDL[CFL], vpFLC via: $\langle a^n \rangle [b^n] p$ ineffable in PDL[CFL]
Also separates PDL[VPL] and vpFLC

Have already seen this in vpFLC: $(\mu X. \langle a \rangle ; [b] \vee \langle a \rangle ; X ; [b]) ; p$

Witness for: $\langle a^n \rangle [b^n] p$ ineffable in PDL[CFL]



Top LTS satisfies $\langle a^n \rangle [b^n] p$, bottom one does not

k, l to be determined

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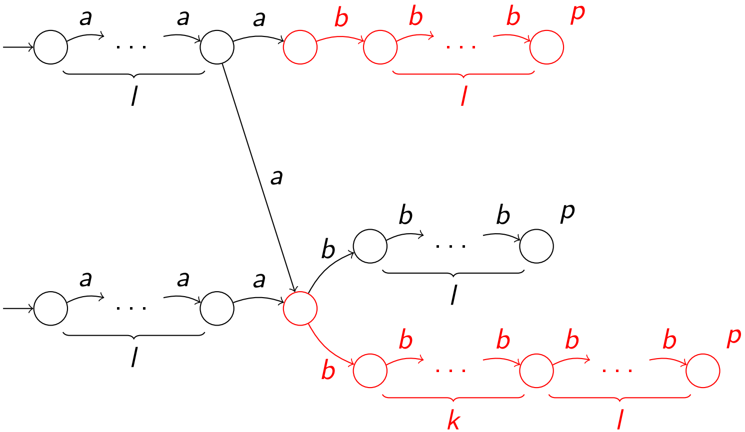

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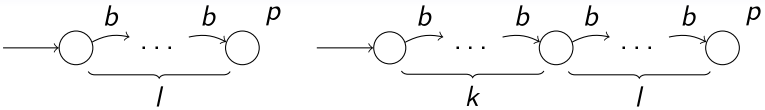
Analyzing the Difference



Assume: φ separates the two LTS; Fix:

- CFL $L_1, \ldots, L_n$ used in φ
- modal depth d (nesting depth of $\langle \cdot \rangle$)

k, l still open, but > 0

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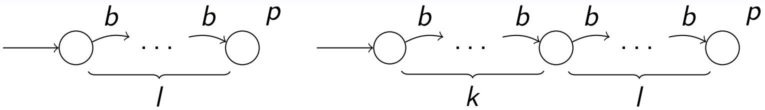

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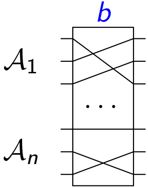
- Need to get rid of the length difference
- Only one transition label: $\rightarrow$ context-free $=$ regular
  - $L_1, \dots, L_n$ w.l.o.g. regular, given as DFA $\mathcal{A}_1, \dots, \mathcal{A}_n$
- Problem: Several languages to consider, even if regular

$\rightarrow$ simultaneous pumping lemma for regular languages needed

Transition Profiles

Goal: categorize behavior of all DFA $\mathcal{A}_1, \ldots, \mathcal{A}_n$ simultaneously:



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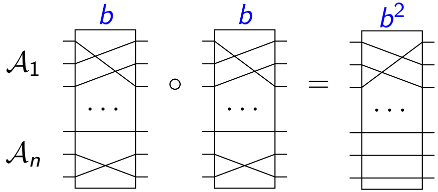

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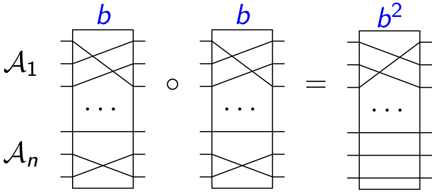
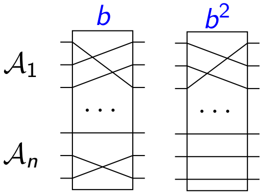


Enumerate profiles for $b, b^2, \dots \rightarrow$ eventually one will repeat



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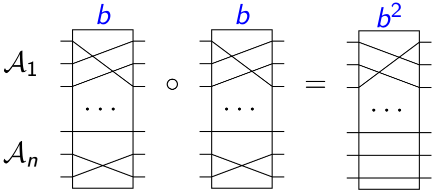

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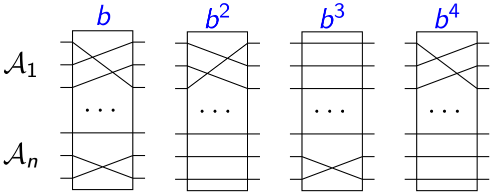

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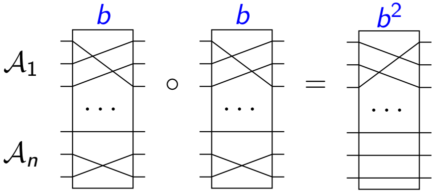

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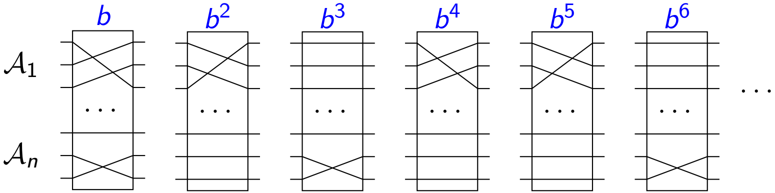

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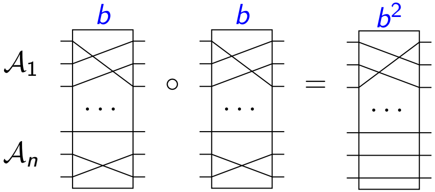

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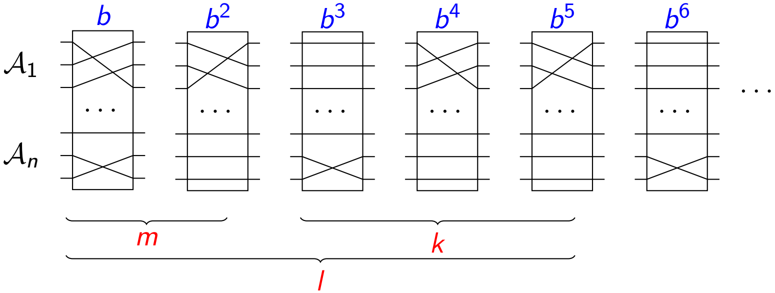

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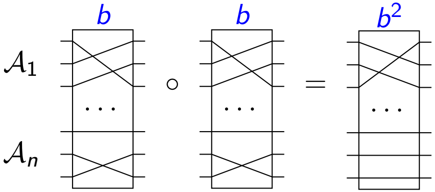

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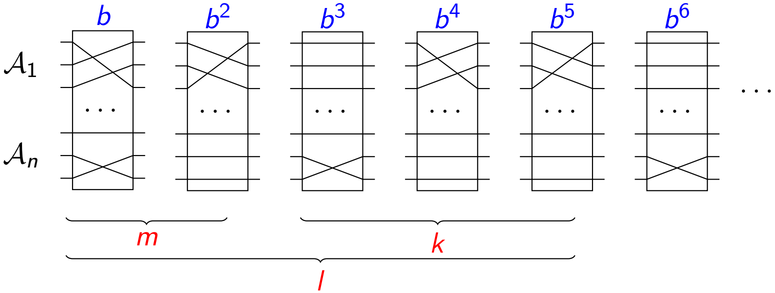
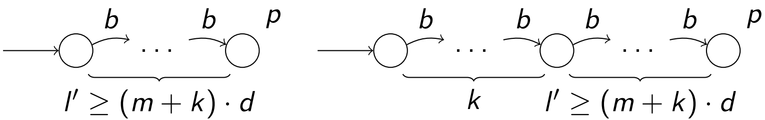


Can pump $b^{l=m+k} \leftrightarrow b^m$ with no DFA "noticing"

Analyzing the Difference (ctd.)

Lifting to modal depth d:

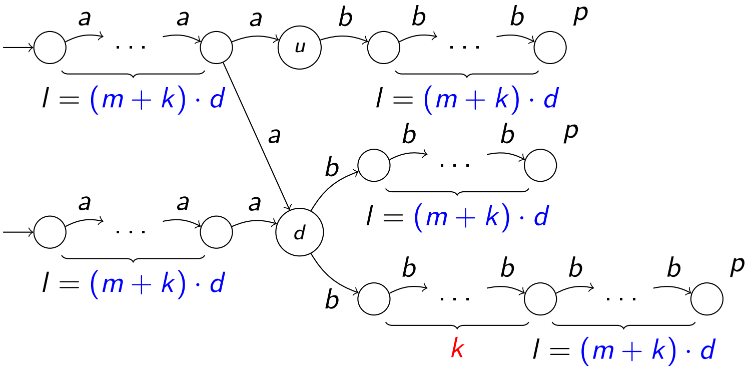


Lemma 1

F.a. CFL $L_1, \ldots, L_n$, modal depth d, ex. $m, k > 0$ s.t. $\varphi \in \mathrm{PDL}[L_1, \ldots, L_n]$ of modal depth d cannot distinguish the above LTS if $l' \geq (m+k) \cdot d$.

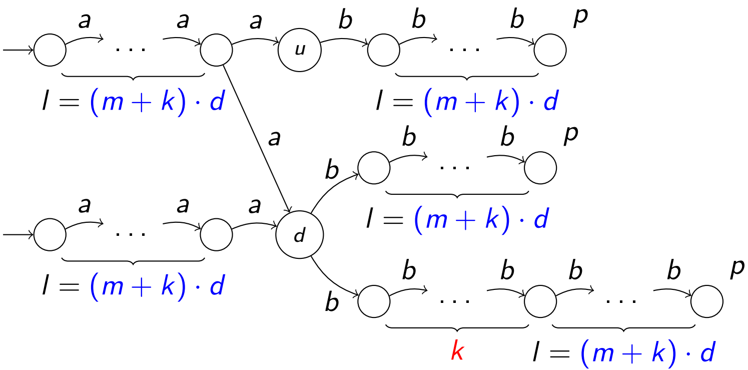
In other words: given $L_1, \ldots, L_n$, and depth d, can find pair of structures to cheat φ

Finishing the Proof



Rest of proof: Induction, case distinction

Finishing the Proof



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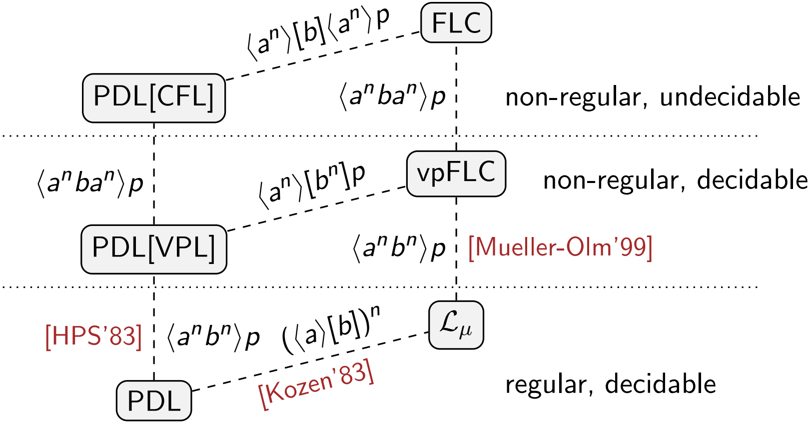
Theorem 2

$\langle a^n \rangle [b^n] p$ *is ineffable in* PDL[CFL] *hence* vpFLC $\not\leq$ PDL[CFL], vpFLC $\not\leq$ PDL[VPL].

Summary



Key points:

- upwards: increase in language power yields separation
- left to right: modality alternation yields separation

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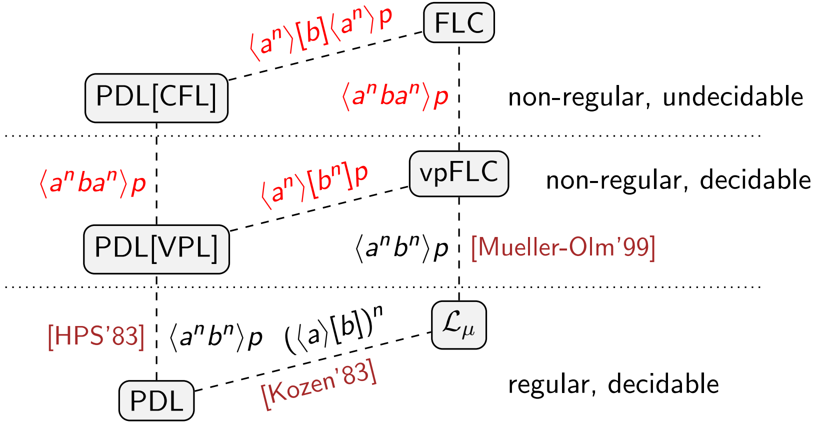

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